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ORDERING PROPERTIES OF LINEAR SUCCESSIVE ITERATION SCHEMES

by

Jack Heller

January 15, 1958

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Heller
Ordering properties of
linear successive
iteration schemes.

institute of mathematical sciences

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AEC Computing and Applied Mathematics Center
Institute of Mathematical Sciences
New York University

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ABSTRACT

The dependence of the asymptotic rate of convergence on the order of correcting linear equations in a successive iteration scheme is studied. Orderings which have the same asymptotic rate of convergence as the order correcting first equation first, second equation second, ..., n^{th} equation last, are found for general linear systems and linear systems which approximate linear elliptic differential equations.

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ORDERING PROPERTIES OF LINEAR SUCCESSIVE ITERATION SCHEMES

Jack Heller

1. Introduction

As digital computers become equipped with larger and larger internal memories, the problem of solving linear systems internally, i.e. without the use of serial input of data, becomes feasible. With this in mind, a series of experiments on the NYU Univac were undertaken by Held and Heller [1] for the solution of the 5 and 13 point elliptic difference schemes. It was shown in these experiments that in the second order line scheme for the 5 point operator, the order of correcting the lines in a successive scheme does not affect the eigenvalues of the error matrix. For the 13 point fourth order elliptic difference equation however it was shown that the order in which one corrects the lines is critical.

In this paper we will develop a general theory of ordering for successive iteration schemes. Since the matrix elements of the linear systems considered here are arbitrary, all the ordering properties found will apply to general regions, general linear extrapolation (over-relaxation), and general boundary conditions. It will be found that the independence of the eigenvalues of the error matrix for the 5 or 9 point Laplace difference scheme does not depend on making special assumptions about the region of solution or boundary conditions. In the case of the general n^{th} order linear system we will find that any cyclic permutation of the order of successive iteration in the solution of the equations will give rise to error matrices all having the same eigenvalues. Finally, in the case of the fourth order successive line schemes, the number of orderings giving rise to the same eigenvalues for the corresponding error matrices will be enumerated. This procedure of enumeration can be applied to any linear successive iteration scheme.

2. The Error Matrix for a Linear Successive Iteration Scheme.

Given a system of linear equations

(1) $$AX = B$$

to be solved by a successive iteration scheme, we split the matrix A into a sum

(2) $$A = R + S$$

where R is in some sense a lower triangular matrix. This split suggests the iteration scheme

(3) $$RX^{(\nu+1)} = B - SX^{(\nu)}$$

where $X^{(\nu)}$ is the ν^{th} iterate, having started with an initial guess $X^{(0)}$. In recent years, techniques for the solution of linear systems by the successive scheme formally given by (3) have received great attention [2,3,4].

Defining the error after the ν^{th} iterate as

(4) $$\varepsilon^{(\nu)} = X - X^{(\nu)}$$

we obtain from (3), (2) and (1):

$$(5) \qquad \varepsilon^{(\nu+1)} = - R^{-1} S\varepsilon^{(\nu)}$$

The matrix $-R^{-1}S$ is called the error matrix, and its eigenvalues satisfy the determinental equation

$$(6) \qquad \det |\lambda R + S| = 0 \quad .$$

Implied in the splitting of the matrix A of (1) is a given order of correcting the elements of the vector $x^{(\nu+1)}$. Since R is lower triangular, the first component of $x^{(\nu+1)}$ is computed first. Then the second component of $x^{(\nu+1)}$ is computed using the corrected first component. This procedure is continued until all the components of $x^{(\nu+1)}$ are computed. If we choose another ordering in which x_1 is not corrected first, x_2 not second, etc., and new values are used whenever available, we will still have a successive iteration scheme but the error matrices will differ. In general, therefore, we would not expect the eigenvalues to be the same. We will refer to the scheme given by (3) as the "usual" scheme. All others will simply be referred to as "ordered" schemes.

Let $A = (a_{ij})$. Then from (6) we have

$$(7) \quad \lambda R+S = \underline{\Lambda} = \begin{pmatrix} \lambda a_{11} & a_{12} & \cdots & a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ \lambda a_{n-1,1} & \cdots & \lambda a_{n-1,n-1} & a_{n-1,n} \\ \lambda a_{n,1} & \cdots & \lambda a_{n,n-1} & \lambda a_{nn} \end{pmatrix}$$

If we have another ordering, say the i_1^{st} equation corrected first, the i_2^{nd} equation second, etc., then at the $\nu+1^{st}$ iteration, the i_1^{st} equation is

$$(8) \qquad a_{i_1 i_1} x_{i_1}^{(\nu+1)} = - \sum_{j \neq i_1} a_{i_1 j} x_j^{(\nu)} + b_{i_1}$$

the i_2^{nd} equation is

$$(9) \qquad a_{i_2 i_2} x_{i_2}^{(\nu+1)} + a_{i_1 i_2} x_{i_1}^{(\nu+1)} = - \sum_{j \neq i_1, i_2} a_{i_2 j} x_j^{(\nu)} + b_{i_2}$$

and in general

$$(10) \qquad \sum_{\substack{n=1 \\ p,s=1,2,\ldots,n \\ p=s}}^{N} a_{i_p i_n} x_{i_s}^{(\nu+1)} = - \sum_{\substack{n,j=1 \\ j \neq i_n, i_{n-1}, \ldots, i_1}}^{N} a_{i_n j} x_j^{(\nu)} + b_{i_n}$$

If we define

$$(11) \qquad \xi(i_n, j) = \begin{cases} 1; & i_n - j \geq 0 \\ 0; & i_n - j < 0 \end{cases}$$

Then the error matrix $\sigma R+S$ becomes

$$(12) \qquad \sigma R+S = \left(\begin{array}{ccc} \sigma a_{11} & & \\ & \sigma a_{22} & \quad \left\{\sigma(1-\xi(j,i_n))+\xi(j,i_n)\right\} a_{j i_n} \\ \left\{\sigma(1-\xi(i_n,j))+\xi(i_n,j)\right\} a_{i_n j} & & \\ & & \sigma a_{nn} \end{array} \right)$$

From (11) and (12) we see that if an element a_{ij} is multiplied by σ its transpose a_{ji} is not. Setting

$$(13) \qquad \sigma(1-\xi(i_n,j)+\xi(i_n,j)) = \sigma^{1-\tau_{i_n,j}}$$

where

$$(14) \qquad \tau(i_n,j) = \begin{cases} 1; & i_n-j \geq 0 \\ 0; & i_n-j < 0 \end{cases}$$

we have

Theorem I. In a successive iteration scheme for the solution of a linear system, if the order of correcting the equations is $i_1,i_2,\ldots,i_n$, the error matrix has the form

$$(15) \; \Sigma = \begin{pmatrix} \sigma a_{11} & \sigma^{\tau_{12}}a_{12} & & \cdots & \sigma^{\tau_{1n}}a_{1n} \\ \sigma^{1-\tau_{12}}a_{21} & \sigma a_{22} & \sigma^{\tau_{23}}a_{23} & \cdots & \sigma^{\tau_{2n}}a_{2n} \\ \vdots & & \ddots & & \vdots \\ \sigma^{1-\tau_{1,n-1}}a_{n-1,n} & \cdots & \sigma a_{n-1,n-1} & & \sigma^{\tau_{n-1,n}}a_{n-1,n} \\ \sigma^{1-\tau_{1n}}a_{n1} & \cdots & \sigma^{1-\tau_{n-1,n}}a_{n,n-1} & & \sigma a_{nn} \end{pmatrix}$$

We look upon the error matrices (7) and (15) as Lambda-matrices. Then the maximum absolute value (spectral radius) of λ and σ obtained by setting the determinant of these matrices equal to zero, determines the rate of convergence of the iteration scheme[3]. If $|\lambda|_{max} = |\sigma|_{max}$, the rates of convergence of any ordered scheme and the usual scheme will be the same. Two questions present themselves:

1. Under what conditions does $|\lambda|_{max} = |\sigma|_{max}$?
2. To determine min $(|\lambda|_{max}, |\sigma|_{max})$; $|\lambda|_{max} \neq |\sigma|_{max}$.

Question 1 is comparatively simple, since we need make no assumptions about the elements a_{ij} of the linear system. On the other hand, to answer question 2, we need some knowledge of the a_{ij}. We will be content here to answer the first question.

Let us consider the following transformation of an error matrix Σ.

$$S\Sigma S^{-1} = \begin{pmatrix} 1 & & & \\ & \sigma^{\tau_{12}} & & 0 \\ & & \sigma^{\tau_{12}+\tau_{23}} & \\ 0 & & & \sigma^{\tau_{12}+\cdots+\tau_{n-1,n}} \end{pmatrix} \Sigma \begin{pmatrix} 1 & & & & \\ & \sigma^{-\tau_{12}} & & & 0 \\ & & \sigma^{-(\tau_{12}+\tau_{23})} & & \\ 0 & & & \ddots & \\ & & & & \sigma^{-(\tau_{12}+\cdots+\tau_{n-1,n})} \end{pmatrix}$$

$$\begin{pmatrix}
\sigma a_{11} & a_{12} & \sigma^{\Gamma_{13}} a_{13} & & - - - & \sigma^{\Gamma_{1n}} a_{1n} \\
\sigma a_{21} & \sigma a_{22} & a_{23} & \sigma^{\Gamma_{24}} a_{24} & - - - & \sigma^{\Gamma_{2n}} a_{2n} \\
\sigma^{1-\Gamma_{13}} a_{31} & & & & & \\
\vdots & & & & \sigma a_{n-1,n-1} & a_{n,n-1} \\
\sigma^{1-\Gamma_{1n}} a_{n1} & & & & & \sigma a_{nn}
\end{pmatrix}$$

where

(17) $$\Gamma_{ij} = \tau_{ij} - \sum_{k=1}^{j-1} \tau_{k,k+1}$$

Setting the determinants $\sum$ and $S \sum S^{-1}$ equal to zero will give rise to identical values of σ. We see, therefore, that if $\Gamma_{ij} = 0$, the error matrix of an ordered scheme (15) and the usual scheme (7) will have the same eigenvalues. In the case $\Gamma_{ij} \neq 0$, the eigenvalues of $\sum$ and $S \sum S^{-1}$ would be equal only for special values of a_{ij}, e.g. $a_{ij} = 0$. Therefore for general, a_{ij}, we have

Theorem II. For any successive linear iteration scheme with arbitrary a_{ij} the eigenvalues of the error matrix will be the same as the eigenvalues of the corresponding usual scheme, if and only if

(18) $$\tau_{ij} - \sum_{k=1}^{j-1} \quad {}_{k,k+1} = 0 \qquad i < j = 2,3,4,\ldots,n$$

for

(19) $$\tau_{ij} = 0 \text{ or } 1.$$

3. Orderings of General n^{th} Order Systems having the Same Eigenvalues as the Usual Ordering.

To find those orderings whose error matrix has the same eigenvalues as that of the usual ordering, we need only solve (18) with condition (19) and find which orderings these solutions determine. The system (18) is a set of equations whose coefficients are integers and whose solutions are integers. Linear systems of this type arise in the study of linear graph theory and combinatorial topology [5,6]. The fact that this type of equation arises here is not surprising. The form of the error matrix for a general ordering circumvents the problem of keeping track of corrected and uncorrected values, which in terms of fitting the corrected and uncorrected values into a given mesh, can be viewed as a problem in combinatorial topology. We will not make any use of the general theory, but will simply solve the special equations(18)[1].

From (15) we see that the number of unknowns τ_{ij} counting along diagonals is

$$(20) \qquad N_u = (n-1) + (n-2) + \ldots + 2 + 1 = \frac{n(n-1)}{2}$$

From (16) we see that the number of equations is

---

[1] By a solution of (18) we will mean a solution for which all $\tau_{ij} = 0$ or 1.

$$(21) \qquad N_e = (n-2) + (n-3) + \ldots + 2 + 1 = \frac{(n-1)(n-2)}{2}$$

Hence

$$(22) \qquad N_u - N_e = n - 1 \; .$$

Therefore, since the equations are homogeneous, there are always non-trivial solutions of (18) and we need only find those for which $\tau_{ij} = 0$ or 1. We also note that there can be at most one none zero term in the sum $\sum_{k=1}^{j-1} \tau_{k,k+1}$.

Let us consider the equations for $i = 1$.

$$(23) \qquad \begin{aligned} &\tau_{13} - (\tau_{12} + \tau_{23}) = 0 \\ &\tau_{14} - (\tau_{12} + \tau_{23} + \tau_{34}) = 0 \\ &\qquad\vdots \\ &\tau_{1n} - (\tau_{12} + \ldots + \tau_{n-1,n}) = 0 \end{aligned}$$

In the equation for $j = n$, the term $(\tau_{12} + \ldots + \tau_{n-1n})$ must be either 0 or 1. If it is 0, all the other τ_{ij} terms are zero because each bracket in the preceeding equations of (23) is 0. Also all τ_{ij} are zero for all terms $(\tau_{ii+1} + \tau_{i+1i+2} + \ldots + \tau_{n-1n})$ and hence all $\tau_{ij} = 0$. The ordering corresponding to this solution is the usual ordering.

If the sum term in (23) for the equation $j = n$ is not 0, one of the τ_{ii+1} terms must be 1, say $\tau_{\ell\ell+1} = 1$. Then the remaining equations in (23) are determined, for in each equation of (23) the sum term is 0 or 1 and so the first term of the equation is either 0 or 1. The same is true for $i = 2$, since all the sum terms are determined and the remaining term in the equation is then determined. This same argument then applies to $i = 3,4,\ldots,n$. We thus see that when one of the terms in the sum term of (23) for $i = 1$ is chosen as 1, all other τ_{ij} are determined and are different for each choice of $\tau_{\ell\ell+1}$. Since we can choose the $\tau_{\ell\ell+1}$ term in the sum term of (23) for $i = 1$ in $n-1$ ways, we have n different solutions counting the trivial solution $\tau_{ij} = 0$. We thus have the

Lemma: There are but n solutions of

$$\tau_{ij} - \sum_{k=1}^{j-1} \tau_{kk+1} = 0 \qquad i < j = 2,3,\ldots,n$$

for which

$$\tau_{ij} = 0 \text{ or } 1.$$

To see which orderings correspond to these solutions, consider $\tau_{\ell\ell+1} = 1$. Then we have, by observing the form of (23) for all i,

$$
\begin{array}{lll}
\tau_{1j} = 0 & j < \ell & \\
\phantom{\tau_{1j}} = 1 & j \geq \ell & \\
 & & \\
\tau_{2j} = 0 & j < \ell & 2 < \ell \\
\phantom{\tau_{2j}} = 1 & j \geq \ell & \\
\vdots & \vdots & \vdots \\
\tau_{ij} = 0 & j < \ell & i < \ell \\
\phantom{\tau_{ij}} = 1 & j \geq \ell &
\end{array}
$$

and if $i \geq \ell$ $\tau_{ij} = 0$.

The error matrix (15) for this solution is

$$
\begin{pmatrix}
\sigma a_{11} & a_{12} & & - - - & & a_{1\ell-1} & \sigma a_{1} & \sigma a_{1\ell+1} & - - - & & \sigma a_{1n} \\
\sigma a_{21} & \sigma a_{22} & a_{23} & - - - & & a_{2\ell-1} & \sigma a_{2} & - - - & & & \sigma a_{2n} \\
\vdots & & & & & & & & & & \vdots \\
\sigma a_{\ell-1\,1} & - - - & & & \sigma a_{\ell-1\,\ell-2} & \sigma a_{\ell-1\,\ell-1} & \sigma a_{\ell-1} & & & & \sigma a_{\ell-1\,n} \\
a_{\ell 1} & - - - & & & a_{\ell\ell-2} & a_{\ell\ell-1} & \sigma a_{\ell\ell} & a_{\ell\ell+1} & - - - & & a_{\ell n} \\
a_{\ell+1\,1} & - - - & & & a_{\ell+1\,\ell-2} & a_{\ell+1\,\ell-1} & \sigma a_{\ell+1\,\ell} & \sigma a_{\ell+1\,\ell+1} & a_{\ell+1\,\ell+2} & - - - & a_{\ell+1\,n} \\
a_{\ell+2\,1} & - - - & & & a_{\ell+2\,\ell-2} & a_{\ell+2\,\ell-1} & \sigma a_{\ell+2\,\ell} & \sigma a_{\ell+2\,\ell+1} & \sigma a_{\ell+2\,\ell+2} & - - - & a_{\ell+2\,n} \\
\vdots & & & & & & & & & & \\
a_{n\ell} & - - - & & & a_{n\ell-2} & a_{n\ell-1} & \sigma a_{n\ell} & - - - & & & \sigma a_{nn}
\end{pmatrix}
$$

The equations that can be corrected first for some ordering are those which have σ on the diagonal term of the error matrix only; the next equations that can be corrected are those which have a σ on the diagonal term of the error matrix and a σ in the column i_1 equal to the first equation corrected; ..., for the σ terms appear with the corrected values. We thus see from (24) that the order of correcting the equations corresponding to one of the solutions of (18) is

$$\ell,\ \ell+1,\ \ell+2,\ \dots,\ n,\ 1,\ 2,\ \dots,\ \ell-1.$$

Since this is true for any ℓ, we have

Theorem III. Only cyclic permutations of the ordering of a linear system solved by a successive iteration scheme will give rise to the same eigenvalues of their corresponding error matrices.

This result is not surprising. If we started with the ℓ^{th} equation and corrected the $\ell, \ell+1, \dots, n$ equations and then started the iteration scheme as a usual scheme, it would be tantamount to starting with a different initial guess, and, as is well known, the eigenvalues of the error matrix are independent of the initial guess. In the case of special linear systems derived from differential equations, the orderings are not always obvious, and so we will proceed to the study of these cases.

4. Ordering Properties of Linear Difference Schemes Derived from Elliptic Differential Equations Solved by Special Block Iteration Schemes.

Linear systems derived from elliptic differential equations have a systematic form to their matrix; this form is exhibited by the uniformity of the zero and non-zero elements. For example, the difference approximation to $\frac{d}{dx}(D \frac{d}{dx})y = f(x)$ would lead -- in the simplest approximation -- to a linear system of the form

$$(25) \quad \begin{pmatrix} a_{11} & a_{12} & 0 \cdots & & & \\ a_{21} & a_{22} & a_{23} & 0 \cdots & & \\ 0 & a_{32} & a_{33} & a_{34} & 0 \cdots & \\ & & & \ddots & & \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

The a_{ij} depend on the mesh spacing and D. For higher order schemes in one dimension, there will simply be more diagonal rows without zeros. In two dimensions line schemes derived from 2p^th order elliptic differential equations give rise to linear systems whose matrix is of the form

$$\begin{pmatrix}
A_{11} & A_{12} & & & A_{1p+1} & 0 & & & & \\
A_{21} & A_{22} & A_{23} & \cdots & A_{2p+1} & A_{2p+2} & 0 & & & \\
\vdots & & & & & & & & & \\
A_{p+11} & A_{p+12} & & & & A_{p+1p+1} & & A_{p+12p+1} & 0 & \\
0 & A_{p+22} & & & & & A_{p+2p+2} & & A_{p+22p+2} & 0 \cdots
\end{pmatrix}$$

where the A_{ii} are matrices of the coefficients relating points along the line and A_{ij} $(i \neq j)$ are matrices of the coefficients relating points off the line. The matrices A_{ij} are square matrices with the number of rows equal to the number of points along the line.

As in Section 2 we see that the form of the error matrix for a successive scheme is

$$\begin{pmatrix}
\sigma A_{11} & \sigma^{\tau_{12}} A_{12} & - - - - - & \sigma^{\tau_{1p+1}} A_{1p+1} & 0 & - - - & 0 \\
\sigma^{1-\tau_{12}} A_{21} & \sigma A_{22} & \sigma^{\tau_{23}} A_{23} & - - - & \sigma^{\tau_{2p+2}} A_{2p+2} & 0 & - - - \\
 & & & & & & \\
\sigma^{1-\tau_{1p+1}} A_{p+11} & - - - & \sigma^{1-\tau_{pp+1}} \sigma A_{p+1p+1} & \sigma^{\tau_{p+1p+2}} A_{p+1p+2} & & - - - & \\
0 & \ddots & & \ddots & & \ddots & \\
\vdots & & \ddots & & \ddots & & \ddots \\
0 & - - - & \ddots & 0 & \sigma^{1-\tau_{np}} A_{pn} & & \sigma A_{nn}
\end{pmatrix}$$

By a similarity transformation similar to (16) except with diagonal elements multiplied by unit matrices of the proper size, we find, that a sufficient condition for the eigenvalues of the error matrix of any ordering to be equal to eigenvalues of the usual ordering is

$$(27) \quad \tau_{ij} - \sum_{k=i}^{p} \tau_{kk+1} = 0 \qquad i < j = 2,3,\ldots,\begin{Bmatrix} p+i \\ n \end{Bmatrix}_{\min} .$$

We again argue as before that this condition is necessary also, because the A_{ij} are general and not zero and we have the counterpart of theorem II:

Theorem II'. For any successive linear iteration scheme, where the error matrix is of the form given by (26), the eigenvalues of the error matrix will be the same as the eigenvalues of the error matrix of the corresponding usual scheme, if and only if

$$(28) \quad \tau_{ij} - \sum_{k=i}^{p} \tau_{kk+1} = 0 \qquad i < j = 2,3,\ldots,\begin{Bmatrix} p+i \\ n \end{Bmatrix}_{\min.}$$

for

$$(29) \quad \tau_{ij} = 0 \text{ or } 1.$$

The only effect of having the triangles of zeros in the upper right and lower left portion of the error matrix is to limit the number of equations for τ_{ij}. By counting along the diagonals we see that the number of unknown τ_{ij} s is

(30) $$N_u^{(p)} = (n-1) + (n-2) + \dots + (n-p) = np - \frac{p(p+1)}{2}$$

and the number of equations by counting along the diagonal of the similarity transformed error matrix is

(31) $$N_e^{(p)} = (n-2) + (n-3) + \dots + (n-p) = np-(n-1)- \frac{p(p+1)}{2}.$$

The excess of unknowns over equations is

(32) $$N_u^{(p)} - N_e^{(p)} = n - 1 ,$$

which is the same as the case of the full matrix. Because (28) is a homogeneous system with more unknowns than equations, there are always non-trivial solutions.

Whether there are solutions for $\tau_{ij} = 0$ or 1 can be settled by straight forward exhibition, and even the number of different solutions can be obtained in the form of a generating function. However, the number of solutions of the τ_{ij} equations (28) is not equal to the number of orderings, for there is not a one to one correspondence between them. We can easily construct a simple example to show this multi-valued property of the orderings as a function of the τ_{ij} solutions.

If we consider the case of $n = 5$ and $p = 2$, we find that the solutions of (28,29) are

	τ_{12}	τ_{13}	τ_{23}	τ_{24}	τ_{34}	τ_{35}	τ_{45}
(a)	1	1	0	0	0	0	0
(b)	1	1	0	1	1	1	0
(c)	1	1	0	0	0	1	1
(d)	0	1	1	1	0	0	0
(e)	0	1	1	1	0	1	1
(f)	0	0	0	1	1	1	0
(g)	0	0	0	0	0	1	1
(h)	0	0	0	0	0	0	0

Corresponding to these solutions we have the following error matrices (where we have suppressed the A_{ij} which are irrelevant in the present context) and their corresponding orderings

(a) orderings (b) orderings

$$\begin{pmatrix} \sigma & \sigma & 0 & 0 & 0 \\ 1 & \sigma & 1 & 1 & 0 \\ 1 & \sigma & \sigma & 1 & 1 \\ 0 & \sigma & \sigma & \sigma & 1 \\ 0 & 0 & \sigma & \sigma & \sigma \end{pmatrix} \quad \begin{matrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 4 & 5 & 5 \end{matrix} \qquad \begin{pmatrix} \sigma & \sigma & \sigma & 0 & 0 \\ 1 & \sigma & 1 & \sigma & 0 \\ 1 & \sigma & \sigma & \sigma & \sigma \\ 0 & 1 & 1 & \sigma & 1 \\ 0 & 0 & 1 & \sigma & \sigma \end{pmatrix} \quad \begin{matrix} 5 & 5 \\ 3 & 2 \\ 4 & 4 \\ 1 & 1 \\ 2 & 3 \end{matrix}$$

(c) (d)

$$\begin{pmatrix} \sigma & \sigma & \sigma & 0 & 0 \\ 1 & \sigma & 1 & 1 & 0 \\ 1 & \sigma & \sigma & 1 & \sigma \\ 0 & \sigma & \sigma & \sigma & \sigma \\ 0 & 0 & 1 & 1 & \sigma \end{pmatrix} \quad \begin{matrix} 5 & 4 & 5 & 4 \\ 1 & 1 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 5 & 4 & 5 \\ 2 & 2 & 1 & 1 \end{matrix} \quad \begin{pmatrix} \sigma & 1 & \sigma & 0 & 0 \\ \sigma & \sigma & \sigma & \sigma & 0 \\ 1 & 1 & \sigma & 1 & 1 \\ 0 & 1 & \sigma & \sigma & 1 \\ 0 & 0 & \sigma & \sigma & \sigma \end{pmatrix} \quad \begin{matrix} 4 & 2 & 2 & 3 & 3 \\ 5 & 5 & 4 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & 2 \\ 3 & 4 & 5 & 5 & 4 \end{matrix}$$

(e)

$$\begin{pmatrix} \sigma & 1 & \sigma & 0 & 0 \\ \sigma & \sigma & \sigma & \sigma & 0 \\ 1 & 1 & \sigma & 1 & \sigma \\ 0 & 1 & \sigma & \sigma & \sigma \\ 0 & 0 & 1 & 1 & \sigma \end{pmatrix} \begin{array}{cc} 4 & 3 \\ 5 & 5 \\ 2 & 2 \\ 3 & 4 \\ 1 & 1 \end{array}$$

(f)

$$\begin{pmatrix} \sigma & 1 & 1 & 0 & 0 \\ \sigma & \sigma & 1 & \sigma & 0 \\ \sigma & \sigma & \sigma & \sigma & \sigma \\ 0 & 1 & 1 & \sigma & 1 \\ 0 & 0 & 1 & \sigma & \sigma \end{pmatrix} \begin{array}{ccc} 3 & 2 & 2 \\ 4 & 3 & 4 \\ 5 & 5 & 5 \\ 1 & 1 & 1 \\ 2 & 4 & 3 \end{array}$$

(g)

$$\begin{pmatrix} \sigma & 1 & 1 & 0 & 0 \\ \sigma & \sigma & 1 & 1 & 0 \\ \sigma & \sigma & \sigma & 1 & \sigma \\ 0 & \sigma & \sigma & \sigma & \sigma \\ 0 & 0 & 1 & 1 & \sigma \end{pmatrix} \begin{array}{ccc} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \\ 1 & 3 & 2 \end{array}$$

(h)

$$\begin{pmatrix} \sigma & 1 & 1 & 0 & 0 \\ \sigma & \sigma & 1 & 1 & 0 \\ \sigma & \sigma & \sigma & 1 & 1 \\ 0 & \sigma & \sigma & \sigma & 1 \\ 0 & 0 & \sigma & \sigma & \sigma \end{pmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} .$$

We see that every solution of the τ_{ij} equations (28), except the trivial solution which corresponds to the usual ordering, gives rise to more than one ordering. There are two questions to be asked: (1) how many different τ_{ij} solutions are there? and (2) how many orderings correspond to a given τ_{ij} solution?

Before going on to these general questions, there is one case of interest that we can easily answer: the case of $p=1$ [10]. This is the case in which the group of equations being corrected is connected to only two neighboring groups of equations; e.g. the line schemes for the five and nine point Laplace difference operator in two dimensions and the plane schemes for the seven and fifteen point Laplace difference operator in three dimensions.

In the case of $p=1$ the number of unknowns from (30) is $N_u^{(1)} = n-1$ and the number of equations from (31) is $N_e^{(1)} = 0$. Since there are no restrictive τ_{ij} equations, we could choose the first equation to be corrected in any one of n ways, the second in any one of $n-1$ ways, ... We therefore have

Theorem IV. If the error matrix of a linear successive iteration scheme is a triple diagonal form

$$\begin{pmatrix} \sigma A_{11} & \sigma^{\tau_{12}} A_{12} & 0 \cdots & & & \\ \sigma^{1-\tau_{12}} A_{21} & \sigma A_{22} & \sigma^{\tau_{23}} A_{23} & 0 \cdots & & \\ 0 & \sigma^{1-\tau_{23}} A_{32} & \sigma A_{33} & \sigma^{\tau_{34}} A_{34} & 0 \cdots & \\ \vdots & & \vdots & & & \\ 0 & \cdots & & 0 & \sigma^{1-\tau_{nn-1}} A_{nn-1} & \sigma A_{nn} \end{pmatrix},$$

the number of ways in which the equations can be ordered and give rise to error matrices which have the same eigenvalues as the usual ordering is $n!$.

5. The Number of Different τ_{ij} Solutions for a General p.

Let us first consider the case of $p=2$, which corresponds to equations coupled in groups of five, e.g. the line schemes for the bi-harmonic equation in two dimensions. The τ_{ij} equations are

$$(33) \qquad \tau_{13} - (\tau_{12} + \tau_{23}) = 0$$

$$(34) \qquad \tau_{24} - (\tau_{23} + \tau_{34}) = 0$$

$$\vdots$$

$$(35) \qquad \tau_{ii+2} - (\tau_{ii+1} + \tau_{i+1i+2}) = 0$$

$$(36) \qquad \tau_{n-2n} - (\tau_{n-2n-1} + \tau_{n-1n}) = 0 \quad .$$

The equations are coupled via one τ_{ij} from one equation to the next, namely the τ_{i+1i+2}. Therefore a solution of the i^{th} equation will have some bearing on the solution of the $i+1^{st}$ equation.

Let us make the following definitions: let η_i^1 be the number of different solutions for the first i equations with the first term in the bracket of the i^{th} equation equal to one; let η_i^2 have the corresponding meaning for the second term in the bracket; and let σ_i be the number of different solutions for the first i equations with the bracket term of the i^{th} equation equal to zero. We consider the first equation (33). The solutions are

τ_{12}	τ_{13}	τ_{23}
0	1	1
1	1	0
0	0	0

and

$$\eta_1^1 = 1$$

$$\eta_1^2 = 1$$

$$\sigma_1 = 1 \quad .$$

When we consider two equations (33,34), we obtain new solutions if the coupled variable (in this case τ_{23}) is zero, for then the bracket term can either be zero or one giving two possible solutions for each previous solution with $\tau_{23} = 0$. The solutions are, in tabular form

τ_{12}	τ_{13}	τ_{23}	τ_{24}	τ_{34}
0	1	1	1	0
1	1	0	1	1
1	1	0	0	0
0	0	0	1	1
0	0	0	0	0 .

For this case

$$\eta_2^1 = 2$$

$$\eta_2^2 = 2$$

$$\sigma_2 = 2 \; .$$

In general whenever there is a one for the coupled variable no new solutions arise and

(37) $$\eta_i^1 = \eta_{i-1}^2$$

In the i^{th} equation the uncoupled variable of the bracket term can be one for every solution with the coupled variable zero. The coupled variable is zero only when the bracket term of the previous equation is zero or the first term of the previous equation is one. We therefore have

(38) $$\eta_i^2 = \sigma_{i-1} + \eta_{i-1}^1$$

In the i^{th} equation the bracket terms can be zero if the coupled variable is zero. The coupled variable is zero only when the bracket term of the previous equation is zero or the first term of the previous equation is one. We therefore have

(39) $$\sigma_i = \sigma_{i-1} + \eta_{i-1}^1 \; .$$

The number of different solutions for the first i^{th} equations is

(40) $$N_i^{(2)} = \eta_i^1 + \eta_i^2 + \sigma_i$$

because the sum on the right represents all the possible values of the bracket terms of the τ_{ij} (p=2) equations.

We can find a somewhat simpler expression for the number of solutions, if we consider

$$(41) \qquad N^{(2)}_{i+1} - N^{(2)}_{i} = \eta^{2}_{i+1}$$

using (37), (38), (39) and (40). Assuming we knew η^{2}_{i+1}, we could solve (41) as a difference equation:

$$(42) \qquad N^{(2)}_{i} = \text{const} + N^{(2)}_{1} + \sum_{k=2}^{i} \eta^{2}_{k} \ .$$

From the above case of i=1, we see that $N^{(2)}_{1} = 3$. Hence the constant in (42) must vanish and

$$(43) \qquad N^{(2)}_{i} = 3 + \sum_{k=2}^{i} \eta^{2}_{k} = 2 + \sum_{k=1}^{i} \eta^{2}_{k}$$

since $\eta^{2}_{1} = 1$. From equations (37), (38) and (39)

$$(44) \qquad \begin{pmatrix} \eta^{1}_{i} \\ \eta^{2}_{i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta^{1}_{i-1} \\ \eta^{2}_{i-1} \end{pmatrix} \ .$$

Using the values for i=1 and iterating, we have

$$(45) \qquad \begin{pmatrix} \eta^{1}_{i} \\ \eta^{2}_{i} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{i-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ .$$

For a system of n equations, the maximum i=n-2. If we call $N^{(2)}$ the number of different solutions of the τ_{ij} equations for a linear system of n equations, we have

$$(46) \qquad N^{(2)} = 2 + \sum_{k=1}^{n-2} \eta_k^2$$

and

$$(47) \qquad \begin{pmatrix} \eta_{n-1}^1 \\ \eta_{n-1}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (n \geq 3) .$$

In the case of $n=5$, we have $\eta_1^2 = 1$, $\eta_2^2 = 2$, $\eta_3^2 = 3$, and then from (46), $N^{(2)} = 2+1+2+3 = 8$ which verifies the example of Section 4.

We notice an interesting sidelight from (37), (38) and (39):

$$(48) \qquad \eta_i^2 = \eta_{i-1}^2 + \eta_{i-2}^2 .$$

A sequence of terms in which the i^{th} term is the sum of the two previous terms is called a Fibonacci sequence. We here are interested in the sum of a Fibonacci sequence. Series of this type are called recurring series; both Fibonacci sequences and recurring series have received a great deal of attention by mathematicians since their discovery in the early middle ages [8].

In a similar manner we can find a generating function for the number of solutions of the τ_{ij} equations and a general p ($\leq n-1$). We define as before, η_i^ℓ as the number of different solutions for the first i equations and all j with the ℓ^{th} term in the bracket of the i^{th} equation and maximum j equal to one, and σ_i as the number of different solutions for the first i equations and all j with the bracket term of the i^{th} equation and maximum j equal to zero.

Only the solutions for the i^{th} equation and maximum j need be considered, for as we saw before the other equations for this i are completely determined. The coupling of the i^{th} equation for maximum j takes place as before in the bracket term only. The maximum value of j is $i+p+1$ $(\leq n)$. Two successive equations for maximum j are

$$(49)\quad \tau_{ii+p+1}-(\tau_{ii+1}+\tau_{i+1i+2}+\dots+\tau_{i+pi+p+1}) = 0$$

$$(50)\quad \tau_{i+1i+p+2}-(\tau_{i+1i+2}+\dots+\tau_{i+pi+p+1}+\tau_{i+p+1i+p+2}) = 0.$$

The coupling is of the same form as in the case $p=2$; and we easily show by similar arguments that

$$(51)\qquad \left\{\begin{array}{l} \eta_i^1 = \eta_{i-1}^2 \\ \eta_i^2 = \eta_{i-1}^3 \\ \vdots \\ \eta_i^{p-1} = \eta_{i-1}^p \end{array}\right.$$

and

$$(52)\qquad \eta_i^p = \sigma_{i-1} + \eta_{i-1}^1 = \sigma_i$$

which are the counterparts of (37) and (38), and (39).

The number of different solutions for the first i^{th} equations and all j is

$$N_i^{(p)} = \eta_i^1 + \eta_i^2 + \dots + \eta_i^p + \sigma_i \tag{53}$$

because the sum on the right represents all the possible values of the bracket terms of the τ_{ij} equations for maximum j. As before we find that

$$N_{i+1}^{(p)} - N_i^{(p)} = \eta_{i+1}^p \tag{54}$$

and the solution, looked upon as a difference equation for $N_i^{(p)}$ is

$$N_i^{(p)} = p + \sum_{k=1}^{i} \eta_k^p , \tag{55}$$

which satisfies the condition $N_1^{(p)} = p+1$.

Finally, as before, the generating function portion of the solution is

$$\begin{pmatrix} \eta_i^1 \\ \eta_i^2 \\ \vdots \\ \\ \eta_i^p \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ & & & \vdots & & \\ 0 & \dots & & & 0 & 1 \\ 1 & 0 & \dots & & 0 & 1 \end{pmatrix}^{i-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \\ 1 \end{pmatrix} \tag{56}$$

For a system of n equations the maximum $i=n-p$, and the total number of solutions is

$$(57) \qquad N^{(p)} = p + \sum_{k=1}^{n-p} \eta_k^p \qquad (p \leq n - 1)$$

where η_k^p is evaluated by using the generating function (56).

We can sum up the results in the following

Theorem V. The number of different solutions with $\tau_{ij} = 0$ or 1 of the equations

$$(58) \quad \tau_{ij}-(\tau_{i\,j+1}+\tau_{i+1\;i+2}+\dots+\tau_{i+p\;i+p+1}) = 0 \quad i<j=2,3,\dots \left\{\begin{matrix} i+p \\ n \end{matrix}\right\}_{\min.}$$

is

$$(59) \qquad N^{(p)} = p + \sum_{k=1}^{n-p} \eta_k^p \qquad 2 \leq p \leq n - 1$$

where η_k^p is determined from (56).

Since the number of τ_{ij} solutions are a lower bound to the number of orderings for which the eigenvalues of the error matrix are the same as the usual ordering, as can be seen from the example in Section 4, we have

Corollary V. 1. The number of orderings $N_0^{(p)}$ which give rise to the same eigenvalues as the usual ordering satisifes

$$n! \geq N_0^{(p)} \geq N^{(p)} \qquad 2 \leq p \leq n - 1 .$$

From (59), when $p=n-1$, the sum term reduces to $\eta_1^{n-1} = 1$, and hence we have the

Corollary V. 2. The number of different solutions with $\tau_{ij} = 0$ or 1 of the τ_{ij} equations for the full matrix is

$$N^{(n-1)} = n \quad .$$

This, of course, is the result that we found in Section 3.

The interesting sidelight we found for $p=2$ has its generalization, for we obtain from (51) and (52) that

$$\eta_i^p = \eta_{i-1}^p + \eta_{i-p}^p \quad (i > p) \quad , \tag{60}$$

which defines a type of recurring series [8].

6. The Number of Orderings whose Error Matrix has the Same Eigenvalues as the Usual Ordering.

As we saw in Section 4, each solution of the τ_{ij} equations gave rise in general to more than one possible ordering. The problem of counting the number of orderings equivalent to the usual ordering for a general p seems very difficult from the present point of view. As we have not been able to solve this problem, we will content ourselves in showing why the problem is so difficult.

We first give a classification to the solutions of the τ_{ij} equations (28-9). By substitution,

$$\tau_{ij} = \phi_i - \phi_j \tag{61}$$

is a solution of (28). If we normalize the ϕ_i solutions, since only the difference is relevant, by setting $\phi_1 = 1$, all remaining ϕ_i can be determined when the τ_{ij} are given. First note that since $\tau_{ij} = 0$ or 1, $\max(\phi_j) \leq 1$.

In order to find the $\min(\phi_i)$, consider the following array

$$\begin{array}{lcccccc}
\text{Line } 1 & \phi_1-\phi_2 & \phi_1-\phi_3 & \cdots & \phi_1-\phi_{p+1} & & \\
\text{Line } 2 & 1 & \phi_2-\phi_3 & \cdots & \phi_2-\phi_{p+1} & \phi_2-\phi_{p+2} & \\
\text{Line } 3 & & 1 & \cdots & & \phi_3-\phi_{p+2} & \phi_3-\phi_{p+3} \\
\vdots & & & & & & \vdots \\
\text{Line } n-1 & & & & & 1 & \phi_{n-1}-\phi_n \\
\text{Line } n & & & & & & 1
\end{array}$$

This is the array of exponents of the upper half of the error matrix. Since $\phi_1 = 1$ and $\tau_{ij} = 0$ or 1, $\phi_2, \phi_3, \ldots, \phi_{p+1} = 0$ or 1.

To find $\min(\phi_j)$, start on the first line of the array and decrease ϕ_j consistent with the requirement that $\phi_i - \phi_j = 0$ or 1. On line 1 $\phi_1 = 1, \phi_2 = \phi_3 = \ldots = \phi_{p+1} = 0$. On line 2 ϕ_2 through ϕ_{p+1} is determined by the choice on line 1. However we can choose ϕ_{p+2}; and the smallest value it can take on is $\phi_{p+2} = -1$. On line 3 all ϕ_i's are determined by lines 2 and 1 except ϕ_{p+3} which must be chosen as $\phi_{p+3} = -1$. We are forced to choose the one undetermined ϕ_i by the previous lines as -1 until we arrive at line p+2. All ϕ_i on this line are determined except ϕ_{2p+2}. Since $\phi_{p+1} - \phi_{2p+2} = 0$ or 1 and since $\phi_{p+1} = -1$, the $\min(\phi_{2p+2}) = -2$. This process can be continued until all the rows are exhausted.

We see that the minimum length of a string of constant ϕ_j is p, since the undetermined ϕ_j on a line could have been set equal to the other ϕ_j's. Define $[n/p]$ as the largest integer contained in n/p, then

$$(62) \qquad \min(\phi_j) = -\left[\frac{n}{p}\right] + 1$$

since this is the number of minimum strings that can be formed from n considering the first and last ϕ_j to be a string of minimum length. For example if $n=6$ and $p=2$, the values of ϕ_j which produce $\min(\phi_j)$ are

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6
1	0	0	-1	-1	-2 .

We can summarize these results in the following

Lemma: A representation of the solution of the τ_{ij} equations (28) is

$$\tau_{ij} = \phi_i - \phi_j$$

and if $\phi_1 = 1$,

(a) $\max(\phi_j) = 1$.

(b) ϕ_j is a decreasing integer valued function and

(c) $\min(\phi_j) = 1 - [\frac{n}{p}]$.

This lemma affords us a classification of solutions according to the number of strings of constant ϕ_j. In the example of Section 4 where $n=5$ and $p=2$ we have

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	Solution label of Section 4.
1	0	0	0	0	(a)
1	1	0	0	0	(d)
1	1	1	0	0	(f)
1	1	1	1	0	(g)
1	1	1	1	1	(h)
1	0	0	-1	-1	(b)
1	0	0	0	-1	(c)
1	1	0	0	-1	(e)

The first group of solutions (a), (d), (f) and (g) will be referred to as one jump solutions; the second group of solutions (h) will be referred to as no jump solutions; the third group of solutions (b), (c) and (e) will be referred to as two jump solutions. The number of jumps is equal to the number of constant strings. Since this classification seems reasonably natural, we will try to count the number of orderings given by each type of ϕ_i solution. Let us restrict ourselves to the case of p=2.

Consider one jump at ϕ_j

$$\begin{array}{ccccc} \phi_{j-1} & \phi_j & \phi_{j+1} & \phi_{j+2} & \cdots \\ 1 & 1 & 0 & 0 & \cdots \end{array}$$

and the corresponding error matrix, where the A_{ij} which are irrelevant in the present context, are suppressed.

$$(63)\quad \begin{array}{c} \\ j-1 \\ j \\ j+1 \\ j+2 \\ \\ \end{array} \left(\begin{array}{ccccccccccccc} & & \ddots & \ddots & \ddots & \ddots & & & & & & & \\ 0 & \cdots & 0 & \sigma & \sigma & \sigma & 1 & \sigma & 0 & \cdots & & & \\ 0 & \cdots & & 0 & \sigma & \sigma & \sigma & \sigma & \sigma & 0 & \cdots & & \\ 0 & \cdots & & & 0 & 1 & 1 & \sigma & 1 & 1 & 0 & \cdots & \\ & & & & & 0 & 1 & \sigma & \sigma & 1 & 1 & 0 & \cdots \\ & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \end{array} \right)$$

Designate e_j as the j^{th} equation and let the symbols $e_i < e_j$ mean equation i is corrected before equation j; then a one jump ϕ_i solution implies the ordering

(64) $e_1 < e_2 < \ldots < e_j$

(65) $e_{j+1} < e_{j-1}$

(66) $e_{j+2} < e_j$

(67) $e_{j+1} < e_{j+2} < \ldots < e_n$

as can be easily seen from the form of the error matrix given by (63); e.g. the j^{th} equation cannot be corrected until the $j+2^{nd}$ equation is corrected which is the meaning of (66).

The problem of counting the number of equivalent orderings has been re-expressed; i.e. to counting the number of arrangements of the symbols e_j subject to constraints of the form (64-67). When there are two jumps to the $\emptyset_i$ solution there will be two sets of constraints:

(68) $e_1 < e_2 < \ldots < e_{j_1}$

(69) $e_{j_1+1} < e_{j_1-1}$

(70) $e_{j_1+2} < e_{j_1}$

(71) $e_{j_1+1} < e_{j_1+2} < \ldots < e_{j_2}$

(72) $e_{j_2+1} < e_{j_2-1}$

(73) $e_{j_2+2} < e_{j_2}$

(74) $e_{j_2+1} < e_{j_2+2} < \ldots < e_n$.

Even though the solutions can be typed in a fairly neat way, it seems impossible to count the number of equivalent orderings.

In the case of one jump (64-67) it is possible to count the number of equivalent orderings. Let us consider the following urn problem: How many ways can j white balls and $n-j$ black balls be arranged in n urns? This problem is a simple problem in combinatorial analysis whose answer is the coefficient $\alpha^j\beta^{n-1}$ in the expansion of $(\alpha+\beta)^n$. Looking at one of these arrangements and calling the first white ball e_1, the second white ball e_2, ...; and the first black ball e_{j+1}, the second black ball e_{j+2}, ...; we would have an arrangement satisfying (64) and (67). Some of the arrangements would satisfy (65) and (66) and some would not. If we could now enumerate the arrangements that do not satisfy (65) and (66) and subtract them from the urn problem count, we would have those arrangements that would satisfy the restrictions (64-7). Summing this over j from 1 to $n-1$, we would have the total number of orderings specified by the one jump ϕ_i solutions. Going to the two jump ϕ_i solutions (68-74), we could consider the urn problem of distributing j_1 white balls, j_2-j_1 black balls and $n-j_2$ orange balls. The number of arrangements is given by the proper coefficient of the trinomial expansion

$(\alpha+\beta+\gamma)^n$. Some of these arrangements will violate the conditions (69), (70), (72) and (73). Proceeding as before we would find the number of orderings specified by the two jump ϕ_i solutions. Continuing on this way, we would try to enumerate all the orderings. The success of this approach depends on the enumeration of the exceptions to the simple urn problems described above. The solution of the urn problems is simple, because we need only find the proper coefficient of an expansion of a symmetric function [9]. The exceptions (65), (66) and (69), (70), (72), (73) seem not be be countable by a coefficient of an expansion of a symmetric function and therein lies the difficulty.

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